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Periodic motions of vortices on surfaces with symmetry

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The theory of point vortices in a two-dimensional ideal fluid has a long history, but on surfaces other than the plane no method of finding periodic motions (except relative equilibria) of N vortices is known. We present one such method and find infinite families of periodic motions on surfaces possessing certain symmetries, including spheres, ellipsoids of revolution and cylinders. Our families exhibit bifurcations. N can be made arbitrarily large. Numerical plots of bifurcations are given.

1. Introduction

Since the 19th century (Helmholtz 1858; Kirchhoff 1876), the Hamiltonian theory of point vortices in a two-dimensional ideal fluid has been a classic chapter in hydrodynamics (cf. Aref 1983). Besides being a useful mathematical prototype of fluid motion, point vortices are of direct physical interest because they can be desingularized to solutions of the Euler equation.

Much is known about periodic motions of these vortices on the plane. On surfaces other than the plane, however, the subject is less well explored. On spheres, Pekarsky & Marsden (1998) investigated the stability and Lim, Montaldi & Roberts (2001) the classification of relative equilibria (vortices that move as a rigid configuration), whereas on cylinders, flat tori and spheres, Aref & Stremler (1996), Stremler & Aref (1999) and Kidambi & Newton (1998, 1999, 2000*a*, *b*) carried out extensive studies of three vortices. Apart from these pioneering works, few results pertaining to N vortices on general surfaces existed until recently; in particular, practically no examples of periodic motion were known. The first infinite family of periodic motions, *dancing vortices*, was found by Tokieda (2001) (and independently worked out by James Montaldi). But dancing vortices were all of a single topological type, i.e. as the parameters varied, they did not undergo bifurcation.

In the present paper we construct an infinite bifurcating family of periodic motions of vortices on surfaces possessing discrete symmetries. N can be made arbitrarily large. The case of dihedral symmetry, including spheres, ellipsoids of revolution and cylinders, is described in detail: the *twisters* of § 3. In § 5, the cases of cyclic and regular polyhedral symmetries are outlined.

Where should we look for such a family of periodic motions? The idea is to endow solutions with so much symmetry that they are left with only two degrees of freedom, and in the resulting two-dimensional system use the Hamiltonian to delineate closed curves. This is reminiscent of the standard approach to integrable systems, which reduces dimensions by finding independent, Poisson-commuting first integrals. An advantage of our approach is that we need not find these first integrals, since we are concerned with individual trajectories, not with invariant tori throughout the phase space; a possible disadvantage is that perturbation theory for symmetric solutions like ours may require a delicate book-keeping of how dynamics intertwines with symmetry.

In plane theory, a sizable literature is already available on periodic motions involving symmetries, e.g. Aref (1982), Koiller *et al.* (1985), Lewis & Ratiu (1996); notably Aref (1982) constructed, from a different viewpoint, a plane analogue of our twisters. Hally (1980) appears to be the earliest to treat vortex motion on surfaces with symmetry.

A good variation on our model would be to add a slight thickness to the two-dimensional fluid, so that we could examine, for instance, bending of vortex tubes and the evolution of an ensemble of Ekman boundary layers. Tropical cyclones (cf. e.g. Holton 1979) constitute a class of meteorological objects into which our theory, or its variant, is likely to provide some qualitative insight. These cyclones are intense vortical storms over tropical oceans above regions of warm surface water (exceeding say 26 °C); they are each typically a couple of hundred kilometres across and are born as a cluster extending up to thousands of kilometres, a scale over which the curvature of the Earth begins to be felt. Another area of application is the study of a rotating layer of liquid helium II (cf. Patterson 1974).

2. Point vortices on surfaces

We recall the theory and list explicit formulae for Hamiltonians on the plane as well as on several other surfaces. Consider an ideal fluid flow in the complex plane $\mathbb{C}$, irrotational except in a domain D bounded by a loop $\gamma = \partial D$. The velocity field $\boldsymbol{v}$ is generated by a stream function ψ. Shrink D to a point z while keeping the circulation $\Gamma = \int_\gamma \boldsymbol{v} \cdot \mathrm{d}\gamma$ constant; in the limit, $\nabla \times \boldsymbol{v} = -\nabla^2\psi$ converges to Γ times a δ-function concentrated at z, an object called *plane point vortex* of *vorticity* Γ. The dynamics of N interacting vortices lends itself to a Hamiltonian formalism: the phase space $\mathbb{C}^N \backslash \{z_k = z_l \text{ for } k \neq l\}$, the direct-sum Kähler metric weighted with vorticities $\sum_{k=1}^N \Gamma_k \,\mathrm{d}z_k \,\mathrm{d}\bar{z}_k$ and the Hamiltonian $h = -(1/2\pi) \sum_{k<l} \Gamma_k \Gamma_l \log |z_k - z_l|$ which is the weighted sum of N Green's functions ψ for N vortices. Indeed, in the plane $\mathbb{C}$ a fundamental solution to $-\nabla^2\psi = \delta$ is $\psi \propto \log r$, where r denotes the Euclidean distance between vortices. The equation of motion is

$$\frac{\mathrm{d}z_k}{\mathrm{d}t} = \frac{2}{\mathrm{i}} \frac{\partial h}{\partial(\Gamma_k \bar{z}_k)} = -\frac{1}{2\pi\mathrm{i}} \sum_{l \neq k} \frac{\Gamma_l}{\bar{z}_k - \bar{z}_l} \quad (k = 1, \ldots, N);$$

this means that each of its partners z_l drags the vortex z_k with velocity proportional to Γ_l, inversely proportional to $|z_k - z_l|$, sweeping counter-clockwise if $\Gamma_l > 0$ and clockwise if $\Gamma_l < 0$. For example (figure 1) when $N = 4$ and $\Gamma_1 = -\Gamma_2 = \Gamma_3 = -\Gamma_4$, if initially the vortices are placed in order at the corners of a rectangle, then they move symmetrically along the branches of the curve $x^{-2} + y^{-2} = \text{constant} > 0$. The dancing vortices mentioned in § 1 are based on this example; it will be used again in § 5.

On the sphere S^2, the cylinder $S^1 \times \mathbb{R}$, the hyperbolic plane H^2, the flat torus T^2 and orientable surfaces in general, the Hamiltonian vortex formalism is defined similarly by calculating the Laplacian ∇^2 and solving for its Green's function.

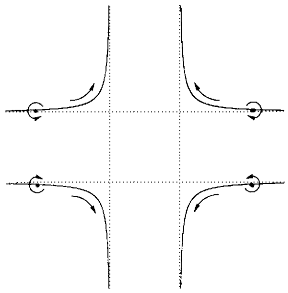

FIGURE 1. Motion of vortices of vorticity $\Gamma_1 = -\Gamma_2 = \Gamma_3 = -\Gamma_4$ on the plane.

On the unit sphere S^2,

$$h = -\frac{1}{4\pi} \sum_{k<l} \Gamma_k \Gamma_l \log(1 - \cos r_{kl}),$$

where r_{kl} denotes the spherical distance between vortices with labels k and l. It should be noted here that $\nabla^2 \log(1 - \cos r) \propto \delta - 4\pi$ and not δ; we are obliged to impose a *background vorticity* $-4\pi = -\text{area}(S^2)$, for by Stokes's formula the integral of $\nabla \times \boldsymbol{v}$ over any closed surface must vanish. Since on account of its isotropy the background vorticity does not affect the vortex dynamics, we shall ignore it in what follows.

On the cylinder $S^1 \times \mathbb{R}$ of unit radius, the Hamiltonian is obtained by formally periodizing the plane Hamiltonian (invoke $\sin z = z \prod_{n>1}(1 - z^2/n^2\pi^2)$) and then jettisoning additive constants to force the convergence:

$$h = -\frac{1}{2\pi} \sum_{k<l} \Gamma_k \Gamma_l \log \left| \sin \frac{z_k - z_l}{2} \right|,$$

where $z = \phi + \mathrm{i}\theta$, and ϕ (modulo 2π), θ are the coordinates on the horizontal circle S^1, the vertical generator $\mathbb{R}$.

On the hyperbolic plane H^2, $\psi \propto \log \tanh(r/2)$, where r this time denotes the hyperbolic distance; on the torus T^2 and on an ellipsoid of revolution, ψ is expressible in terms of a Jacobian theta-function and Lamé harmonics respectively. The Hamiltonians are weighted sums of these ψ.

Naturally, the theories on all these surfaces reduce to the theory on the plane in the limit of infinitesimal distances.

3. Periodic motions with dihedral symmetry

3.1. *Twisters on the sphere*

On the sphere S^2, we take as coordinates the longitude ϕ (modulo 2π) and the latitude θ ($-\pi/2 < \theta < \pi/2$) (rather than the usual colatitude in spherical coordinates). Consider $2N + 2$ vortices positioned as in figure 2(a):

N vortices of vorticity $+1$ at $(\phi, \theta), (\phi + 2\pi/N, \theta), \ldots, (\phi + (N-1)2\pi/N, \theta)$,

N vortices of vorticity $+1$
at $(-\phi, -\theta), (-\phi - 2\pi/N, -\theta), \ldots, (-\phi - (N-1)2\pi/N, -\theta)$,

2 vortices of vorticity Γ at the north and south poles.

Vortices in this configuration, or *twisters*, form a parametric family, with a discrete parameter N and a continuous parameter Γ. At all times, twisters are invariant

FIGURE 2. (*a*) Coordinates for twisters on the sphere. (*b*) Generators of D_{2N}.

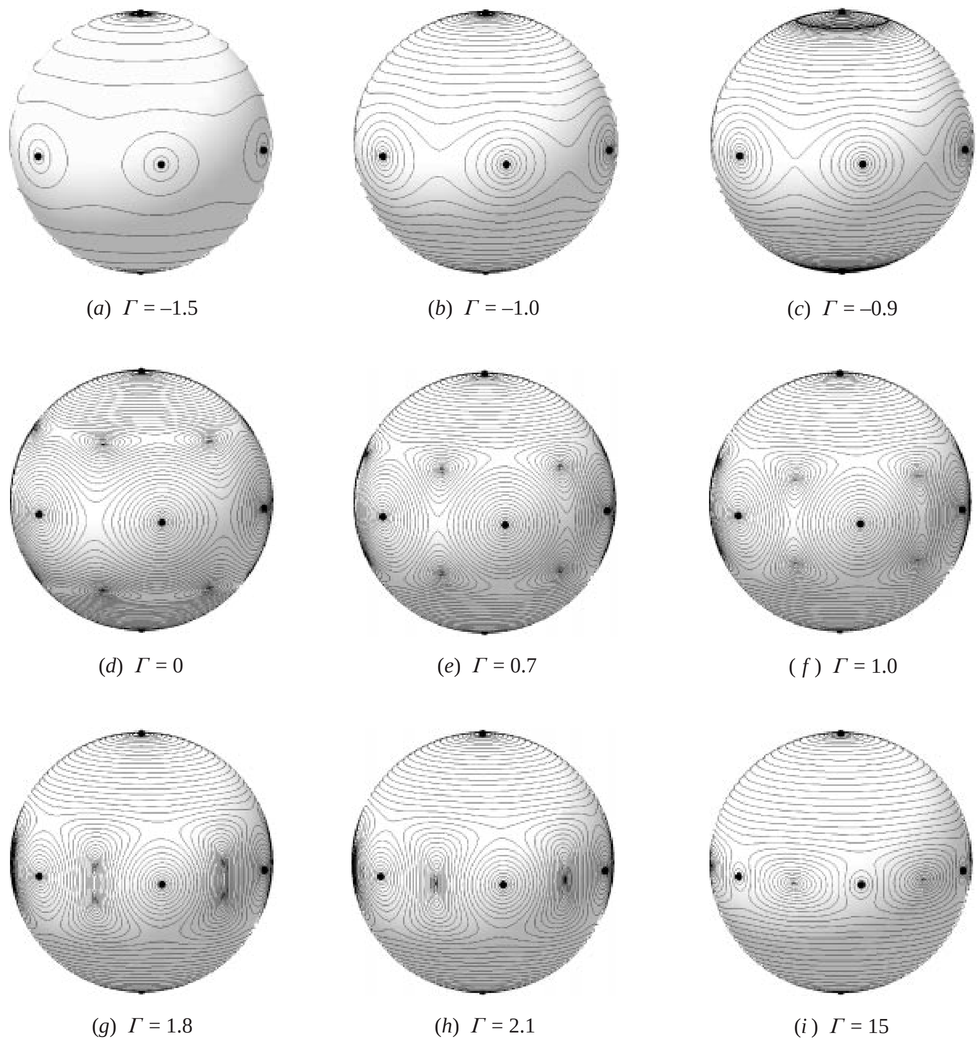

FIGURE 3. Twister on the sphere, N=3.

under the action of the dihedral group D_{2N}: this group is generated by the rotation $(\phi, \theta) \mapsto (\phi + 2\pi/N, \theta)$ around the pole–pole axis and the flip $(\phi, \theta) \mapsto (-\phi, -\theta)$ around the Gulf of Guinea–Kiribati axis (figure 2*b*). (The choice of $+1$ as the value of vorticity is immaterial, but the $2N$ non-polar vortices must have the same vorticity if the symmetry D_{2N} is to act.) Thanks to these symmetries, the positions of all $2N+2$ twisters are determined by the position of any non-polar one of them. Let us designate one non-polar vortex as a *tracer* and draw its trajectory on the phase sphere (ϕ, θ). Figure 3(a–i) shows the result for $N = 3$; the motions of the remaining $2N-1$ non-polar vortices are reconstructed by applying D_{2N} to the tracer; as regards the polar vortices, they never move. The $2N$ black dots on the equator at $(0,0), (\pm\pi/N, 0), \ldots, (\pm(N-1)\pi/N, 0), (\pi, 0)$ represent collision singularities where the northern and southern rings become superposed; the two black dots at the poles represent coalescence of each ring with a polar vortex. The Hamiltonian h diverges at all these points.

While $\Gamma < -(N-1)/2$, there are $2N$ equatorial saddles representing the northern and southern rings in fully staggered positions on the equator. They are connected by separatrices, which separate three regimes of periodic motions: above the separatrices, westward wavy motion; below the separatrices, eastward wavy motion; inside the separatices, counter-clockwise spinning motions around collision singularities.

When $\Gamma = -(N-1)/2$, we see at the poles the births of new regimes of periodic motions which expand as Γ increases $> -(N-1)/2$. Figure 3(c), $\Gamma = -0.9$, shows the new regimes soon after their births, for $N = 3$; by $\Gamma = 0$ (figure 3d) they have grown and are easier to see. On each hemisphere, they are: $2N$ clockwise centres and $2N$ new saddles, the latter being connected by new separatrices; in the northern polar cap bordered by the new separatices, eastward wavy motion; in the southern polar cap, westward wavy motion. Within the equatorial band bordered by the new separatrices the motion is as before.

At a higher value of Γ, the new and old separatrices in the equatorial band merge, squeezing to nil the old westward and eastward wavy motions: figure 3(e), $\Gamma = 0.7$.

From then on, as shown in figure 3(f, g), $\Gamma = 1.0, 1.8$, the new saddles on the two hemispheres become mutually connected by separatrices, whereas every separatrix issuing from an old equatorial saddle circumnavigates a new centre and returns to the same saddle. Just outside the 'figures of eight' formed by the returning separatrices, a new regime of clockwise, peanut-shaped periodic motions appears.

Meanwhile, the 'new' (no longer so new) centres above and below the equatorial saddles have been coming closer together. They eventually merge, as in figure 3(h), $\Gamma = 2.1$, absorbing between them the equatorial saddles. As Γ continues to increase, the equatorial band where interesting things are happening becomes thinner and thinner, and the now overgrown polar caps of eastward and westward wavy motions invade a larger and larger portion of the sphere, but no further bifurcations occur.

3.2. *Twisters on the cylinder*

On the cylinder $S^1 \times \mathbb{R}$, with coordinates specified in §2, consider $2N$ vortices of vorticity $+1$ in the following positions:

$$N \text{ vortices at } (\phi, \theta), (\phi + 2\pi/N, \theta), \ldots, (\phi + (N-1)2\pi/N, \theta),$$
$$N \text{ vortices at } (-\phi, -\theta), (-\phi - 2\pi/N, -\theta), \ldots, (-\phi - (N-1)2\pi/N, -\theta).$$

As in the case of the sphere S^2, these twisters are invariant under the action of the dihedral group D_{2N}. Figure 4 shows the trajectory of a tracer vortex for $N = 3$. As the only parameter at our disposal is the discrete parameter N, we have a single picture for each N and no bifurcation.

3.3. *Twisters on ellipsoids of revolution*

Though the Hamiltonian formalism is difficult to write out explicitly, it is clear that surfaces possessing the symmetries of D_{2N} are always hospitable to periodic

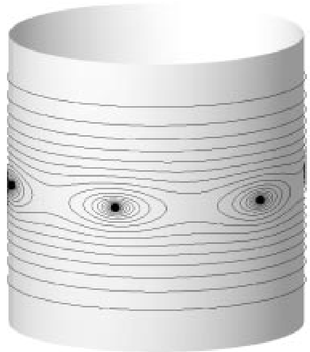

FIGURE 4. Twisters on the cylinder.

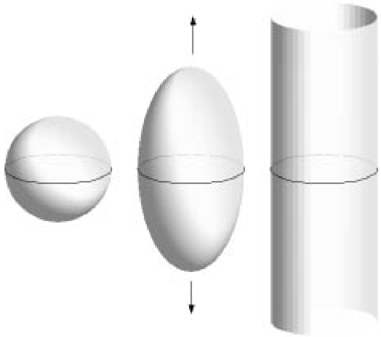

FIGURE 5. Ellipsoids of revolution interpolating between a sphere and a cylinder.

motions of twisters. If the surface is diffeomorphic to a sphere, polar vortices may be introduced, with attendant bifurcations. The case of ellipsoids of revolution deserves special attention, for it can be thought of as an interpolation between the sphere case and the cylinder case (figure 5).

4. Analysis of periodicity, bifurcation and level spacing

4.1. *Proof of periodicity by symmetry and Hamiltonian*

It was observed in § 3 that the positions of twisters are determined by the position of any non-polar one of them. In other words, for given N and Γ, all possible positions of twisters form a certain two-dimensional surface Σ in the phase space; our phase sphere (figure 3) or phase cylinder (figure 4) is a diffeomorphic projection of Σ on the 'tracer component' factor of the phase space. Since the dihedral symmetry D_{2N} preserves the Hamiltonian h, Σ is dynamically invariant. Now h is a first integral: in other words, every level set of h is dynamically invariant, generically a hypersurface of codimension 1 in the phase space. Hence a phase trajectory of twisters lies in the intersection of Σ and a level set of h. Such an intersection is generically a one-dimensional curve, and depending on whether it is compact or not, it is diffeomorphic to a circle S^1 (periodic motion) or to a line $\mathbb{R}$ (separatrix); it can also degenerate to a point (equilibrium). This achieves the proof that the generic orbits are periodic.

To draw the trajectories on the phase sphere or cylinder, it therefore suffices to regard h as a function of two variables ϕ, θ and plot its level curves. In the case of the sphere S^2, by a routine calculation,

$$2^{1-2N-\Gamma^2/N}\mathrm{e}^{-4\pi h/N}$$
$$= (\cos^2\theta)^{2\Gamma+N-1}(1-\cos^2\theta\cos^2\phi)\prod_{k=1}^{N-1}\sin^2\left(\frac{k\pi}{N}\right)\left(1-\cos^2\theta\cos^2\left(\phi+\frac{k\pi}{N}\right)\right).$$

In the case of the cylinder $S^1\times\mathbb{R}$,

$$\mathrm{e}^{-2\pi h/N} = (\sin^2\phi+\sinh^2\theta)\prod_{k=1}^{N-1}\sin\left(\frac{k\pi}{N}\right)\left(\sin^2\left(\phi+\frac{k\pi}{N}\right)+\sinh^2\theta\right).$$

The diffeomorphism $x=\sin\phi$ ($|\phi|<\pi$), $y=\sin\theta$ ($|\theta|<\pi/2$) or $y=\sinh\theta$ ($|\theta|<\infty$) converts the above equations into (almost) algebraic equations and maps trajectories to (almost) algebraic curves. The proviso 'almost' is due to the exponent Γ for S^2: for rational Γ, these are genuine algebraic curves. For $S^1\times\mathbb{R}$ the problem does not arise. A corollary of algebraicity is that degeneracies cannot be too bad: for example, equilibria are isolated (actually mere analyticity implies this conclusion). Sharper information concerning the numbers of ovals (periodic motions in our context), branches (separatrices), isolated points (equilibria) and their arrangements may be obtained from the theory of topology of real algebraic curves, originated by Harnack; cf. Bochnak, Coste & Roy (1987) and the state of the art in Mikhalkin (2000).

The case of ellipsoids of revolution, though analytically harder, is topologically identical to the case of the sphere S^2.

4.2. *Critical polar vorticity*

For twisters on the sphere S^2, the critical value $-(N-1)/2$ of the polar vorticity Γ for the earliest bifurcation (figure 3(*b*), $\Gamma=-1.0$ when $N=3$) may be deduced by comparison with the theory on the plane. In the plane $\mathbb{C}$, a regular N-gon of vortices of vorticity $+1$ spins counter-clockwise. The vorticity of a vortex that needs to be inserted at the centre to immobilize the polygon is $-(N-1)/2$, independently of the scale of the polygon. On S^2, as the northern ring of twisters approaches the north pole, the influence from the southern vortices becomes negligible, while in the neighbourhood of the north pole the sphere theory is approximated better and better by the plane theory. So, in the limit, $\Gamma<-(N-1)/2$ successfully induces a clockwise (westward) motion on the northern hemisphere, whereas $\Gamma>-(N-1)/2$ is insufficient to brake the counter-clockwise (eastward) motion of the ring. Because the motion farther south is westward, the latter scenario leads inevitably to the birth of new equilibria.

4.3. *Logarithmic divergence of the Hamiltonian*

Near the collision and coalescence singularities (e.g. black dots in figure 3), the Hamiltonian h diverges like $-\log\epsilon$, $\epsilon\searrow 0$. This makes the spacing between levels crowded near the singularities (which is not relevant here) and inconveniently sparse near the equilibria (where we wish to zoom in). In numerical simulations, we took the logarithmically biased spacing

$$\min h + (\max h - \min h)\left(\frac{l}{L}\right)^{\beta}\quad (l=0,\ldots,L)$$

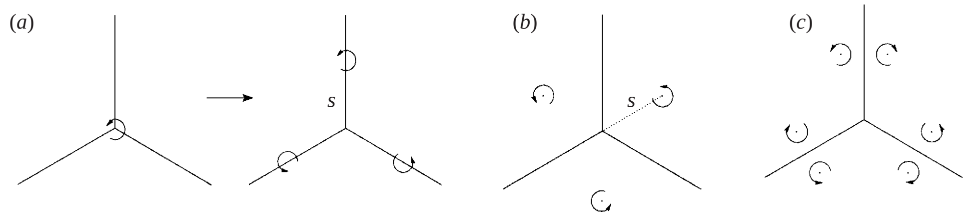

FIGURE 6. Splitting a vortex: (*a*) along the edges; (*b*) along the faces; (*c*) astride the edges, alternating vorticities.

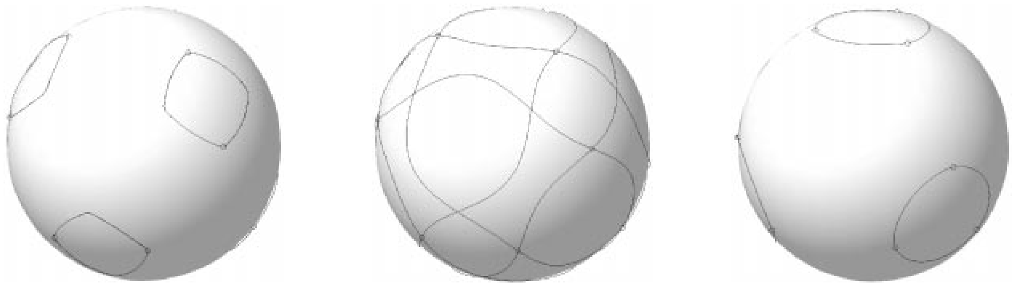

FIGURE 7. Vortices starting on the edges near the vertices of a regular tetrahedron.

(L = number of levels plotted), whose effect is to enhance the resolution near minima if $\beta > 1$ and near maxima if $\beta < 1$. Pictures in figures 3 and 4 were plotted at biases $\beta = 5,\ 1/2$.

5. Periodic motions with other symmetries

Symmetries different from that of the dihedral group D_{2N} can be exploited to construct periodic motions of vortices. We adopt the notation $\mathbb{Z}_N$, S_n, A_n for the cyclic group of order N, the group of all permutations of n objects (symmetric group), the group of even permutations of n objects (alternating group) respectively. $\mathbb{Z}_N$, D_{2N}, A_4, S_4, A_5 exhaust all discrete subgroups of the group of spatial rotations $SO(3)$, which is the spatial symmetry group of the Hamiltonian vortex theory on the sphere S^2. (The theory has a few extra non-spatial symmetries, such as the antipodal map combined with the reversal of vorticities.)

Four vortices of vorticity $+1$ at the vertices of a regular tetrahedron are in equilibrium. Now at each vertex, split the vortex along the edges into three vortices of vorticity $+1/3$ as in figure 6(*a*). The resulting motion of 4×3 vortices (figure 7) is periodic and bifurcates once as the *split parameter s* varies. The symmetry is A_4. We obtain analogous periodic motions with symmetries S_4, S_4, A_5, A_5 starting from a cube, a regular octahedron, a regular dodecahedron, a regular icosahedron. Splitting along the faces as in figure 6(*b*) again creates periodic motions; this time, moreover, as the split parameter varies the vortices of the given polyhedron move like the vortices of its dual polyhedron as the latter's split parameter varies backward.

We have also found periodic motions to which the argument of § 4.1 does not apply. Figure 8 shows two parallel equilateral triangles of three vortices of vorticity $+1$ at asymmetric latitudes. The symmetry group is $\mathbb{Z}_3$.

Note that dancing vortices (cf. figure 1 and Tokieda 2001) too can accommodate symmetries of regular polyhedra. Figure 9 shows the trajectories of $4 \times 3 \times 2$ vortices

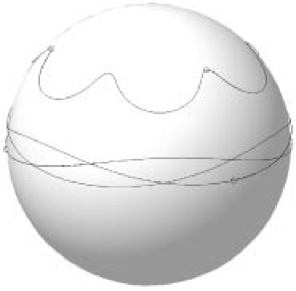

FIGURE 8. Equilateral triangles of vortices at asymmetric latitudes.

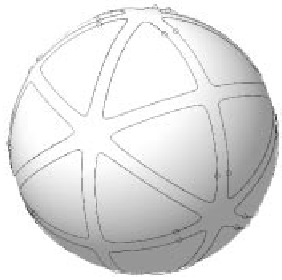

FIGURE 9. Dancing vortices with tetrahedral symmetry.

of alternating vorticities started around the vertices of a regular tetrahedron at the positions of figure 6(*c*).

Let us summarize the approach to the construction of periodic motions of N vortices developed in this paper.

(1) Give the vortices initial positions with a discrete symmetry that preserves the Hamiltonian. Hamilton's equation being first-order ('vortices have no inertia'), the velocities thereby inherit the same symmetry, so that the vortices will keep the same symmetry throughout their evolution.

(2) Estimate the size of the symmetry, or more precisely, count the number G of different group orbits in the given initial positions: the larger the size, the smaller G. The dimension $2N$ of the phase space is reduced to $2G$.

(3) If at least $2G - 1$ independent first integrals are available, then generic trajectories are periodic. Even if there are too few first integrals, an application of the intermediate value theorem may still detect some periodic motions.

Detection of *asymmetric* periodic motions is of course beyond the scope of our approach. Aref & Vainchtein (1998) discovered asymmetric relative equilibria of vortices on the plane.

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